Probability and Statistics in Engineering, Fall 2016

<u>Exercise #3</u>

1. Suppose a special type of small data processing firm is so specialized that some have difficulty making a profit in their first year of operation. The pdf that characterizes the proportion *Y* that make a profit is given by

$$f(y)=\begin{cases} ky^4(1-y)^3, & 0\le y\le 1 \\ 0, & \text{elsewhere} \end{cases}$$

   (a) What is the value of *k* that renders the above a valid density function?
   (b) Find the probability that at most 50% of the firms make a profit in the first year.
   (c) Find the probability that a least 80% of the firms make a profit in the first year.

Sol:

(a) Using integral by parts and setting $1 = k\int_0^1 y^4(1-y)^3\,dy$, we obtain $k = 280$.

(b) For $0 \le y < 1$, $F(y) = 56y^5(1-Y)^3 + 28y^6(1-y)^2 + 8y^7(1-y) + y^8$. So, $P(Y \le 0.5) = 0.3633$.

(c) Using the cdf in (b), $P(Y > 0.8) = 0.0563$.

2. Magnetron tubes are produced from an automated assembly line. A sampling plan is used periodically to assess quality on the lengths of the tubes. This measurement is subject to uncertainty. It is thought that the probability that a random tube meets the length specification is 0.99. A sampling plan is used in which the lengths of 5 random tubes are measured. *Y* is defined as the number of tubes, out of 5, that meet the length specification.
   (a) Find the probability function of *Y*.
   (b) Suppose the result of a sampling plan is that 3 tubes are outside the specification. Use the probability function, obtained in (a), to support or refute the conjecture that the probability is 0.99 that a single tube meets the specification.

Sol:

(a) The event $Y = y$ means that among 5 selected, exactly y tubes meet the specification (M) and $5-y$ (M') does not. The probability for one combination of such a situation is $(0.99)^y(1-0.99)^{5-y}$ if we assume independence among the tubes. Since there are $\frac{5!}{y!(5-y)!}$ permutations of getting y Ms and $5-y$ M's, the probability of this event ($Y = y$) would be what it is specified in the problem.

(b) Three out of 5 is outside of specification means that $Y = 2$. $P(Y = 2) = 9.8\times 10^{-6}$ which is extremely small. So, the conjecture is false.

3. A candy company distributes boxes of chocolates with a mixture of creams, toffees, and cordials. Suppose that the weight of each box is 1 kilogram, but the individual weights of the creams, toffee, and cordials vary from box to box. For a randomly selected box, let *X* and *Y* represent the

weights of the creams and the toffees, respectively. In addition, suppose that the joint density function of these variables is

$$f(x,y)=\begin{cases}24xy, & 0\le x\le 1,\ 0\le y\le 1,\ x+y\le 1\\ 0, & \text{elsewhere}\end{cases}$$

(a) Find the probability that in a given box, the cordials account for more than 1/2 of the weight.

(b) Find the marginal density for the weight of the creams.

(c) Find the probability that the weight of the toffees in a box is less than 1/8 of a kilogram if it is know that the creams constitute 3/4 of the weight.

Sol:

(a) $P(X+Y\le 1/2)=\int_0^{1/2}\int_0^{1/2-y}24xy\,dx\,dy=12\int_0^{1/2}\left(\frac{1}{2}-y\right)^2 y\,dy=\frac{1}{16}$.

(b) $g(x)=\int_0^{1-x}24xy\,dy=12x(1-x)^2$, for $0\le x<1$.

(c) $f(y|x)=\frac{24xy}{12x(1-x)^2}=\frac{2y}{(1-x)^2}$, for $0\le y\le 1-x$.
Therefore, $P(Y<1/8\mid X=3/4)=32\int_0^{1/8}y\,dy=1/4$.

4. The amount of kerosene, in thousands of liters, in a tank at the beginning of any day is a random amount Y from which a random amount X is sold during that day. Suppose that the tank is not resupplied during the day so that $x\le y$, and assume that the joint density function of these variables is

$$f(x,y)=\begin{cases}2, & 0<x<y<1\\ 0, & \text{elsewhere}\end{cases}$$

(a) Determine if X and Y are independent.

(b) Find $P(1/4<X<1/2\,|\,Y=3/4)$.

Sol:

(a) $g(x)=2\int_x^1 dy=2(1-x)$ for $0<x<1$;
$h(y)=2\int_0^y dx=2y$, for $0<y<1$.
Since $f(x,y)\ne g(x)h(y)$, X and Y are not independent.

(b) $f(x|y)=f(x,y)/h(y)=1/y$, for $0<x<y$.
Therefore, $P(1/4<X<1/2\mid Y=3/4)=\frac{4}{3}\int_{1/4}^{1/2}dx=\frac{1}{3}$.

5. Random variables X and Y represent the number of vehicles that arrive at 2 separate street corners during a certain 2-minute period. These street corners are fairly close together so it is important that traffic engineers deal with them jointly if necessary. The joint distribution of X and Y is known to be

$$f(x,y)=\frac{9}{16}\frac{1}{4^{x+y}},\ \text{for } x=0,1,2,\ldots,\ \text{and } y=0,1,2,\ldots$$

(a) Are the two random variables X and Y independent? Explain the reason.

(b) What is the probability that, during the time period in question, less than 4 vehicles arrive at the two street corners?

Sol:

(a) $g(x) = \frac{9}{(16)4^y} \sum_{x=0}^{\infty} \frac{1}{4^x} = \frac{9}{(16)4^y} \frac{1}{1-1/4} = \frac{3}{4} \cdot \frac{1}{4^x}$, for $x = 0, 1, 2, \ldots$; similarly, $h(y) = \frac{3}{4} \cdot \frac{1}{4^y}$, for $y = 0, 1, 2, \ldots$. Since $f(x, y) = g(x)h(y)$, X and Y are independent.

(b) $P(X+Y < 4) = f(0,0) + f(0,1) + f(0,2) + f(0,3) + f(1,0) + f(1,1) + f(1,2) + f(2,0) + f(2,1) + f(3,0) = \frac{9}{16}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{r^2} + \frac{1}{4^3} + \frac{1}{4^3}\right) = \frac{9}{16}\left(1 + \frac{2}{4} + \frac{3}{4^2} + \frac{4}{4^3}\right) = \frac{63}{64}$.

6. (10 pts) Consider a system of components in which there are five independent components, each of which has an operational probability of 0.92. The system has a redundancy built in such that it does not fail if 3 out of the 5 components are operational. What is the probability that the total system is operational?

Sol:

Denote by X the number of components (out of 5) work.
Then, $P(\text{the system is operational}) = P(X \geq 3) = P(X = 3) + P(X = 4) + P(X = 5) = \binom{5}{3}(0.92)^3(1-0.92)^2 + \binom{5}{4}(0.92)^4(1-0.92) + \binom{5}{5}(0.92)^5 = 0.9955$.