5. 數學期望值

Mathematical Expectation

- 隨機變數的平均。
- 隨機變數的變異數與共同變異數。
- 隨機變數線性組合的平均值與變異數。

5.1 隨機變數的平均

- Two coins are tossed 16 times. Suppose that the experiment yields no heads, one head, and two heads a total of 4, 7, and 5 times, respectively. What is the average number of heads per toss of the two coins?

- If one fair coin is tossed twice, what is the average number of heads we would expect?

定義 5.1
已知離散隨機變數 X 以及其機率分佈函數 f(x)，則 X 的平均值(mean)或期望值(expected value)定義為

$$\mu_X = E(X) = \sum_x x f(x)$$

如果 X 是連續隨機變數，則其平均值或期望值為

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

範例5.1

- A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.?

範例5.2

- A salesperson for a medical device company has two appointments on a given day. At the first appointment, he believes that he has a 70% chance to make the deal, from which he can earn $1000 commission if successful. On the other hand, he thinks he only has a 40% chance to make the deal at the second appointment, from which, if successful, he can make $1500. What is his expected commission based on his own probability belief? Assume that the appointment results are independent of each other.

範例5.3

- Let X be the random variable that denotes the life in hours of a certain electronic device. The probability density function is as below. Find the expected life of this type of device.

$$f(x)=\begin{cases}\dfrac{20,000}{x^3}, & x>100\\ 0, & \text{elsewhere}\end{cases}$$

定理 5.1

假設 X 是一個隨機變數且其機率分佈函數為 f(x)，則變數 g(X) 也是一個隨機變數。

當 X 是離散時，g(X) 的期望值定義為 $\mu_{g(X)}=E[g(X)]=\sum_x g(x)f(x)$

當 X 是連續時，g(X) 的期望值定義為 $\mu_{g(X)}=E[g(X)]=\int_{-\infty}^{\infty} g(x)f(x)dx$

From theorem 5.1, the probability function of g(X) is f(x), which is exactly the probability function of X.
Why do they, i.e. g(X) and X, have the same probability function?

範例5.4

- Suppose that the number of cars X that pass through a car wash between 4:00 P.M. and 5:00 P.M. on any sunny Friday has a probability distribution shown below. Let g(X) = 2X−1 represent the amount of money, in dollars, paid to the attendant by the manager. Find the attendant's expected earnings for this particular time period.

x	4	5	6	7	8	9
$P(X=x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

範例5.5

- Let X be a random variable with density function as below. Find the expected value of g(X) = 4X + 3.

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

5.1 隨機變數的平均(contd.)

定義 5.2

令 $f(x,y)$ 是 2 個隨機變數 X 和 Y 的聯合機率分佈，則對隨機變數 $g(X,Y)$ 而言，

若X和Y是離散的，其平均值或期望值為 $\mu_{g(X,Y)} = E[g(X,Y)] = \sum_x \sum_y g(x,y) f(x,y)$

若X和Y是連續的，其平均值或期望值為 $\mu_{g(X,Y)} = E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f(x,y) dxdy$

範例5.6

- Let X and Y be the random variables with joint probability distribution shown below. Find the expected value of g(X, Y) = XY. Also, how to find E(X) and E(Y)?

	$f(x,y)$	x = 0	x = 1	x = 2	Row Totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

範例5.7

- Find E(Y/X) for the density function.

$$f(x,y)=\begin{cases}\frac{x(1+3y^2)}{4}, & 0<x<2, 0<y<1\\ 0, & \text{elsewhere}\end{cases}$$

- If g(X, Y) = X, how to calculate E[g(X, Y)]? How about if g(X, Y) = Y?

5.2 隨機變數的變異數與共同變異數

定義 5.3
已知隨機變數 X 的機率分佈 f(x) 以及其平均值 μ，則隨機變數 X 的變異數為

$$\sigma^2 = E[(X-\mu)^2] = \begin{cases}\sum_x (x-\mu)^2 f(x), & \text{if } X \text{ is discrete}\\ \int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx, & \text{if } X \text{ is continuous}\end{cases}$$

其中，變異數 σ^2的正平方根稱為隨機變數 X 的標準差 (standard deviation)。

範例5.8

- Let the random variable X represent the number of automobiles that are used for official business purposes on any given workday. The probability distributions for companies A and B are shown below. Show that the variance of the probability distribution for company B is greater than that for company A.

A:

x	1	2	3
$f(x)$	0.3	0.4	0.3

B:

x	0	1	2	3	4
$f(x)$	0.2	0.1	0.3	0.3	0.1

定理 5.2

隨機變數 X 的變異數是 $\sigma^2 = E(X^2) - \mu^2$

How to prove it?

Assume that X is discrete

範例5.9

- Let the random variable X represent the number of defective parts for a machine when 3 parts are sampled from a production line and tested. Listed below is the probability distribution of X. Calculate σ^2 using Theorem 5.2.

x	0	1	2	3
$f(x)$	0.51	0.38	0.1	0.01

範例5.10

- The weekly demand for a drinking-water product, in thousands of liters, from a local chain of efficiency stores is a continuous random variable X having the probability density as below. Find the mean and variance of X.

$$f(x) = \begin{cases} 2(x-1), & 1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

5.2 隨機變數的變異數與共同變異數(contd.)

定理 5.3

已知隨機變數 X 之機率分佈函數 f(x)，

若 X 是離散的，則隨機變數 g(X) 的變異數為 $\sigma_{g(X)}^2 = E\{[g(X)-\mu_{g(X)}]^2\} = \sum_x [g(x)-\mu_{g(X)}]^2 f(x)$

若 X 是連續的，則隨機變數 g(X) 的變異數為 $\sigma_{g(X)}^2 = E\{[g(X)-\mu_{g(X)}]^2\} = \int_{-\infty}^{\infty} [g(x)-\mu_{g(X)}]^2 f(x)dx$

範例5.11

- Calculate the variance of g(X) = 2X + 3, where X is a random variable with probability distribution shown below.

x	0	1	2	3
$f(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$

範例5.12

- Let X be a random variable having the density function given in Example 5.5 (page 5). Find the variance of the random variable g(X) = 4X + 3.

定義 5.4

已知隨機變數 X 和 Y 之聯合機率分佈為 $f(x, y)$，則隨機變數 X 和 Y 的共同變異數為

$$\sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = \sum_x \sum_y (x-\mu_X)(y-\mu_Y) f(x, y) \quad , \text{ if } X \text{ and } Y \text{ are discrete}$$

$$\sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y) f(x, y)\,dxdy \quad , \text{ if } X \text{ and } Y \text{ are continuous}$$

定理 5.4

How to prove it?

令 μ_X 和 μ_Y 分別為隨機變數 X 和 Y 的平均值，則 X 和 Y 的共同變異數為 $\sigma_{XY} = E(XY) - \mu_X \mu_Y$

範例5.13

- X and Y have a joint probability distribution shown below. Find the covariance of X and Y.

	$f(x,y)$	$x=0$	$x=1$	$x=2$	$h(y)$
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
	$g(x)$	$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

範例5.14

- The fraction X of male runners and the fraction Y of female runners who compete in marathon races are described by the joint density function as below. Find the covariance of X and Y. Is X + Y = 1?

$$f(x,y)=\begin{cases}8xy\ , & 0\le y\le x\le 1\\ 0\ , & \text{elsewhere}\end{cases}$$

5.2 隨機變數的變異數與共同變異數(contd.)

定義 5.5
假設隨機變數 X 與 Y 的共同變異數為 σ_{XY}、個別的標準差分別為 σ_X 以及 σ_Y 。則隨機變數 X 和 Y的相關係數定義為

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

Does ρ_{XY} have any unit? Why do we need to use ρ_{XY}?

範例5.15

- Find the correlation coefficient between X and Y in Example 5.13?

範例5.16

- Find the correlation coefficient between X and Y in Example 5.14?

5.3 隨機變數之線性組合的平均值與變異數

定理 5.5

如果 a 和 b 都是常數，則 $E(aX+b)=aE(X)+b$ How to prove it?

推論 5.1

令 $a=0$，則 $E(b)=b$。

推論 5.2

令 $b=0$，則 $E(aX)=aE(X)$。

範例5.17

- Applying Theorem 4.5 to the discrete random variable f(X) = 2X−1, rework Example 5.4.

範例5.18

- Applying Theorem 4.5 to the continuous random variable g(X) = 4X+3, rework Example 5.5.

定理 5.6 How to prove it?

兩個隨機變數 X 之函數的和或差，如 $g(X) \pm h(X)$，期望值為 $E[g(X) \pm h(X)] = E[g(X)] \pm E[h(X)]$

範例5.19

- Let X be a random variable with probability distribution shown below. Find the expected value of $Y = (X-1)^2$

x	0	1	2	3
$f(x)$	$\frac{1}{3}$	$\frac{1}{2}$	0	$\frac{1}{6}$

範例5.20

- The weekly demand for a certain drink, in thousands of liters, at a chain of convenience stores is a continuous random variable $g(X) = X^2 + X - 2$, where X has the density function shown below. Find the expected value of the weekly demand for the drink.

$$f(x) = \begin{cases} 2(x-1), & 1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

定理 5.7
隨機變數 X 和 Y 之函數的和或差，如$g(X, Y) \pm h(X, Y)$，期望值為

$$E[g(X, Y) \pm h(X, Y)] = E[g(X, Y)] \pm E[h(X, Y)]$$

How to prove it?

推論 5.3

令 $g(X, Y) = g(X)$ 以及 $h(X, Y) = h(Y)$，我們可得 $E[g(X, Y) \pm h(X, Y)]$
$= E[g(X)] \pm E[h(Y)]$

推論 5.4

令 $g(X, Y) = X$ 以及 $h(X, Y) = Y$，我們可得 $E[g(X, Y) \pm h(X, Y)]$
$= E[X] \pm E[Y]$

5.3 隨機變數之線性組合的平均值與變異數(contd.)

定理 5.8 How to prove it?
令 X 和 Y 為兩個獨立的隨機變數，則 $E(XY)=E(X)E(Y)$

推論 5.5

假設 X 和 Y 為兩個獨立的隨機變數，則 $\sigma_{XY}=0$。

範例5.21

- It is known that the ratio of gallium to arsenide does not affect the functioning of gallium-arsenide (砷化鎵) wafers, which are the main components of microchips. Let X denote the ratio of gallium to arsenide and Y denote the functional wafers retrieved during a 1-hour period. X and Y are independent random variables with the joint density function shown below. Show that E(XY) = E(X)E(Y).

$$f(x,y)=\begin{cases}\dfrac{x(1+3y^2)}{4}, & 0<x<2,\ 0<y<1\\ 0, & \text{elsewhere}\end{cases}$$

5.3 隨機變數之線性組合的平均值與變異數(contd.)

定理 5.9
已知 2 個隨機變數 X 和 Y 的聯合機率分佈為 f(x, y)，假設 a, b, c 為常數，則

$$\sigma^2_{aX+bY+c} = a^2\sigma^2_X + b^2\sigma^2_Y + 2ab\sigma_{XY}$$

How to prove it?

推論 5.6
令 b = 0，則 $\sigma^2_{aX+c} = a^2\sigma^2_X = a^2\sigma^2$

推論 5.7
令 a = 1, b = 0，則 $\sigma^2_{X+c} = \sigma^2_X = \sigma^2$

推論 5.8
令 b = 0, c = 0，則 $\sigma^2_{aX} = a^2\sigma^2_X = a^2\sigma^2$

推論 5.9
如果 X 和 Y 是2個獨立的隨機變數，則 $\sigma^2_{aX+bY} = a^2\sigma^2_X + b^2\sigma^2_Y$

推論 5.10
如果 X 和 Y 是2個獨立的隨機變數，則 $\sigma^2_{aX-bY} = a^2\sigma^2_X + b^2\sigma^2_Y$

推論 4.11
如果 $X_1, X_2, \cdots, X_n$ 為獨立的隨機變數組，則

$$\sigma^2_{a_1X_1+a_2X_2+\cdots+a_nX_n} = a_1^2\sigma^2_{X_1} + a_2^2\sigma^2_{X_2} + \cdots + a_n^2\sigma^2_{X_n}$$

範例5.22

- If X and Y are random variables with variances $\sigma_X^2 = 2$, $\sigma_Y^2 = 4$ and covariance $\sigma_{XY} = -2$, find the variance of the random variable Z = 3X − 4Y + 8.

範例5.23

- Let X and Y denote the amounts of two different types of impurities in a batch of a certain chemical product. Suppose that X and Y are independent random variables with variances $\sigma_X^2 = 2$ and $\sigma_Y^2 = 3$. Find the variance of the random variable Z = 3X − 2Y + 5.

- If g(X) is nonlinear, how to calculate E[g(X)] ?

範例5.24

- Given the random variable X with mean μ_X and variance σ_X^2 , give the second-order approximation to $E(e^X)$.

- If g(X) is nonlinear, how to calculate Var[g(X)] ?

範例5.25

- Given the random variable X as in Example 5.24, give an approximate formula for Var[g(x)].

範例5.26

- Consider two independent random variables X and Z with means μ_X and μ_Z and variances σ_X^2 and σ_Z^2, respectively. Consider a random variable $Y = X/Z$. Give approximations for E(Y) and Var(Y).

5.4 柴比雪夫定理(Chebyshev's Theorem)

定理 4.11

對任何隨機變數 X 而言，已知其平均值 μ_X 與變異數 σ_X^2 ，則該隨機變數 X 的值落在平均值附近的 k 倍標準差內的機率至少是 $1-1/k^2$，亦即 $P(\mu_X - k\sigma_X < X < \mu_X + k\sigma_X) \geq 1 - \frac{1}{k^2}$

When applying Chebyshev's theorem, do we need to know the probability distribution of a random variable?

範例5.27

- A random variable X has a mean $\mu = 8$, a variance $\sigma^2 = 9$, and an unknown probability distribution. Find (a) $P(-4 < X < 20)$, (b) $P(|X - 8| \geq 6)$.