9. 單樣本的估計問題

One-Sample Estimation Problems

- 點估計、區間估計
- 信賴區間、預測區間、容許區間
- 平均值的估計、比例的估計、變異數的估計

9.1 點估計與區間估計

- Statistical inference may be divided into two major areas: estimation（估計）and test of hypothesis (假說檢定).
  - A candidate for public office may wish to estimate the true proportion of voters favoring him by obtaining opinions from a random sample of 100 eligible voters.
  - A person wants to find out whether brand A floor wax is more scuff-resistant than brand B floor wax. He or she might hypothesize that brand A is better than brand B and, after proper testing, accept or reject this hypothesis.
- We are dependent on sampling theory and the use of data to provide us with some measure of accuracy for our decision.

定義 9.1
如果 $\mu_{\hat{\Theta}} = E(\hat{\Theta}) = \theta$，則統計值 $\hat{\Theta}$ 稱為對母體參數 θ 的無偏見估計法 (unbiased estimator)。

範例9.1

- Show that S^2 is an unbiased estimator of the parameter σ^2.

9.1 點估計與區間估計 (contd.)

- For normal populations, one can show that both $\bar{X}$ (樣本平均值) and $\hat{X}$ (樣本中位數) are unbiased estimators of the population mean μ, but the variance of $\bar{X}$ is smaller than the variance of $\hat{X}$.
  - Both estimates $\bar{x}$ and $\hat{x}$ will, on average, equal the population mean μ. But $\bar{x}$ is likely to be closer to μ for a given sample. Thus, $\bar{X}$ is more efficient than $\hat{X}$.

- **區間估計值的詮釋(Interpretation of Interval Estimates)**
  - 樣本中計算出來的區間 $(\hat{\theta}_L, \hat{\theta}_U)$，使得 $P\left(\hat{\theta}_L < \theta < \hat{\theta}_U\right) = 1-\alpha$，稱為 $100(1-\alpha)\%$ 的信賴區間 (confidence interval)，而 $(1-\alpha)$ 稱為信賴係數 (confidence coefficient)，或是信賴程度 (degree of confidence)，區間的端點值則稱為信賴極限 (confidence limit) 的上限和下限。

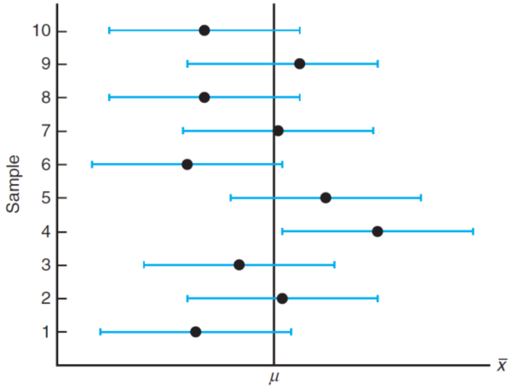
9.2 平均值的估計

假設 σ^2 已知，μ 的信賴區間

假設母體之變異數 σ^2 已知。 若 $\bar{x}$ 是一樣本數為 n 的隨機樣本之平均值，給定 α 且 $0<\alpha<1$，則母體平均值 μ 的 $100(1-\alpha)\%$ 信賴區間為

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

其中 $z_{\alpha/2}$ 代表標準常態分佈曲線下在其右邊的面積為 $\alpha/2$ 的 z-值。



Interval estimates of μ for different samples.

範例9.2

- The average zinc concentration recovered from a sample of measurements taken in 36 different locations in a river is found to be 2.6 grams per milliliter. Find the 95% and 99% confidence intervals for the mean zinc concentration in the river. Assume that the population standard deviation is 0.3 gram per milliliter.

定理 9.1

如果我們用 $\bar{x}$ 來估計 μ，則我們有 $100(1-\alpha)\%$ 信心水準相信其估計誤差不會超過 $z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$。

定理 9.2

What if σ^2 is unknown?

如果我們用 $\bar{x}$ 來估計 μ，則我們有 $100(1-\alpha)\%$ 信心水準相信其估計誤差不會超過一個指定量 e 時所需的樣本大小為 $\left(\frac{z_{\alpha/2}\sigma}{e}\right)^2$。

範例9.3

- How large a sample is required if we want to be 95% confident that our estimate of μ in Example 9.2 is off by less than 0.05?

9.2 平均值的估計 (contd.)

假設σ^2 未知，μ 的信賴區間

假設母體之變異數σ^2 未知。 若 $\bar{x}$ 與 S 分別為一樣本數為 n 的常態母體之隨機樣本的平均值與標準差，給定α 且 $0<\alpha<1$，則母體平均值μ 的$100(1-\alpha)\%$ 信賴區間為

$$\bar{x}-t_{\alpha/2}\frac{s}{\sqrt{n}}<\mu<\bar{x}+t_{\alpha/2}\frac{s}{\sqrt{n}}$$

其中 $t_{\alpha/2}$ 代表自由度為 $n-1$ 之 t-分佈曲線下在其右邊的面積為$\alpha/2$ 的 t-值。

範例9.4

- The contents of seven similar containers of sulfuric acid are 9.8, 10.2, 10.4, 9.8, 10.0, 10.2, and 9.6 liters. Find a 95% confidence interval for the mean contents of all such containers, assuming an approximately normal distribution.?

9.2 平均值的估計 (contd.)

Large-Sample Confidence Interval

Statisticians often recommend that even when normality cannot be assumed and σ is unknown, if n ≥ 30, s can replace σ and the confidence interval $\bar{x} \pm z_{\alpha/2}\frac{s}{\sqrt{n}}$ may be used.

範例9.5

- Scholastic Aptitude Test (SAT) mathematics scores of a random sample of 500 high school seniors in the state of Texas are collected, and the sample mean and standard deviation are found to be 501 and 112, respectively. Find a 99% confidence interval on the mean SAT mathematics score for seniors in the state of Texas.

- The standard error of an estimator is its standard deviation.
- The standard deviation of $\bar{X}$, or standard error (s.e.) of $\bar{X}$, is $\sigma/\sqrt{n}$.

假設 $\boldsymbol{\sigma}^2$ 已知，樣本由常態分佈中隨機取得，$\boldsymbol{\mu}$ 的信賴極限：$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = \bar{x} \pm z_{\alpha/2} \times \text{s.e.}(\bar{x})$

假設 $\boldsymbol{\sigma}^2$ 未知，樣本由常態分佈中隨機取得，$\boldsymbol{\mu}$ 的信賴極限：$\bar{x} \pm t_{\alpha/2}\frac{s}{\sqrt{n}} = \bar{x} \pm t_{\alpha/2} \times \text{s.e.}(\bar{x})$

9.3 預測區間

假設 σ^2 已知，未來觀察值之預測區間(Prediction Interval)

對於一常態分佈而言，假設其平均值 μ 未知但變異數σ^2 已知，則對於**一未來觀察值** x_0 之$100(1-\alpha)\%$ 信心水準下之預測區間為

$$\bar{x} - z_{\alpha/2}\sigma\sqrt{1+1/n} < x_0 < \bar{x} + z_{\alpha/2}\sigma\sqrt{1+1/n}$$

其中 $z_{\alpha/2}$ 為標準常態分佈下在其右邊的面積為 $\alpha/2$ 的 z-值。

How to derive this interval?

範例9.6

- Due to the decrease in interest rates, the First Citizens Bank received a lot of mortgage applications. A recent sample of 50 mortgage loans resulted in an average loan amount of $257,300. Assume a population standard deviation of $25,000. For the next customer who fills out a mortgage application, find a 95% prediction interval for the loan amount.

9.3 預測區間 (contd.)

假設 σ^2 未知，未來觀察值之預測區間

對於一常態分佈而言，假設其平均值 μ 和變異數 σ^2 都未知，則對於**一未來觀察值** x_0 之 $100(1-\alpha)\%$ 信心水準下之預測區間為

$$\bar{x} - t_{\alpha/2} s\sqrt{1+1/n} < x_0 < \bar{x} + t_{\alpha/2} s\sqrt{1+1/n}$$

其中 $t_{\alpha/2}$ 代表自由度為 $n-1$ 之t-分佈下在其右邊的面積為 $\alpha/2$ 的 t-值。

How to derive this interval?

範例9.7

- A meat inspector has randomly selected 30 packs of 95% lean beef. The sample resulted in a mean of 96.2% with a sample standard deviation of 0.8%. Find a 99% prediction interval for the leanness of a new pack. Assume normality.

9.4 容許區間

容許區間 (Tolerance Limits)

對於一平均值 μ 、標準差 σ 都未知之常態分佈的測量值而言，其容許區間 (tolerance interval) 的定義為 $\bar{x} \pm ks$ ，其中 k 為一個常數，k 值的決定是在$100(1-\gamma)\%$信心水準下，其容許區間最少會包含 $(1-\alpha)$ 比例的觀察值。

範例9.8

- Consider Example 9.7. With the information given, find a tolerance interval that gives two-sided 95% bounds on 90% of the distribution of packages of 95% lean beef. Assume the data came from an approximately normal distribution.

Case Study 9.1

- A machine produces metal pieces that are cylindrical in shape. A sample of these pieces is taken and the diameters are found to be 1.01, 0.97, 1.03, 1.04, 0.99, 0.98, 0.99, 1.01, and 1.03 centimeters. Use these data to calculate three interval types and draw interpretations that illustrate the distinction between them in the context of the system. For all computations, assume an approximately normal distribution. The sample mean and standard deviation for the given data are $\bar{x} = 1.0056$ and $s = 0.0246$.

Case Study 9.1 (contd.)

- (a) Find a 99% confidence interval on the mean diameter. (b) Compute a 99% prediction interval on a measured diameter of a single metal piece taken from the machine. (c) Find the 99% tolerance limits that will contain 95% of the metal pieces produced by this machine.

9.5 比例的估計

大樣本數之 p (Proportion) 的信賴區間

如果 $\hat{p}=x/n$ 為一樣本數為 n 之二項式試驗其成功樣本比例，令 $\hat{q}=1-\hat{p}$，則二項式分佈參數 p 的近似 $100(1-\alpha)\%$ 信賴區間為 (方法一)： $\hat{p}-z_{\alpha/2}\sqrt{\hat{p}\hat{q}/n}<p<\hat{p}+z_{\alpha/2}\sqrt{\hat{p}\hat{q}/n}$

或為 (方法二)： $$\frac{\hat{p}+\frac{z_{\alpha/2}^2}{2n}}{1+\frac{z_{\alpha/2}^2}{n}}-\frac{z_{\alpha/2}}{1+\frac{z_{\alpha/2}^2}{n}}\sqrt{\frac{\hat{p}\hat{q}}{n}+\frac{z_{\alpha/2}^2}{4n^2}}<p<\frac{\hat{p}+\frac{z_{\alpha/2}^2}{2n}}{1+\frac{z_{\alpha/2}^2}{n}}+\frac{z_{\alpha/2}}{1+\frac{z_{\alpha/2}^2}{n}}\sqrt{\frac{\hat{p}\hat{q}}{n}+\frac{z_{\alpha/2}^2}{4n^2}}$$

其中 $z_{\alpha/2}$ 是標準常態分佈曲線下右邊機率為 $\alpha/2$ 的 z-值。

To have sufficient reliability, $n\hat{p}$ and $n\hat{q}$ are required to be greater than or equal to 5.

範例9.9

- In a random sample of n = 500 families owning television sets in the city of Hamilton, Canada, it is found that x = 340 subscribe to HBO. Find a 95% confidence interval for the actual proportion of families with television sets in this city that subscribe to HBO.

定理 9.3

如果 $\hat{p}$ 是實際值 p 的一估計值，則我們有 $100(1-\alpha)\%$ 信心水準其誤差值不會超過 $z_{\alpha/2}\sqrt{\hat{p}\hat{q}/n}$ 。

定理 9.4

如果 $\hat{p}$ 是實際值 p 的一估計值，則我們有 $100(1-\alpha)\%$ 信心水準其估計誤差會小於一個指定的數 e，此時樣本數 n 的近似值為

$$n = \frac{z_{\alpha/2}^2 \hat{p}\hat{q}}{e^2}$$

定理 9.5

如果用 $\hat{p}$ 來估計實際值 p，且其樣本數為

$$n = \frac{z_{\alpha/2}^2}{4e^2}$$

則我們最少 (at least) 有 $100(1-\alpha)\%$ 的信心水準相信其誤差值不會超過一個指定的量 e。

範例9.10

- How large a sample is required if we want to be 95% confident that our estimate of p in Example 9.9 is within 0.02 of the true value?

範例9.11

- How large a sample is required if we want to be at least 95% confident that our estimate of p in Example 9.9 is within 0.02 of the true value?

9.6 變異數的估計

$\boldsymbol{\sigma}^2$ 的信賴區間

如果 s^2 是常態母體中隨機挑選之樣本數為 n 的樣本變異數，則母體變異數 $\boldsymbol{\sigma}^2$ 之$100(1-\alpha)\%$的信賴區間為 $\dfrac{(n-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \dfrac{(n-1)s^2}{\chi^2_{1-\alpha/2}}$ ， 其中 $\chi^2_{1-\alpha/2}$ 以及 $\chi^2_{\alpha/2}$ 分別為自由度為 $\upsilon = n-1$ 的開方分佈曲線下

右邊的面積分別為 $1-\alpha/2$ 以及 $\alpha/2$ 的值。

範例9.12

- The following are the weights, in decagrams, of 10 packages of grass seed distributed by a certain company: 46.4, 46.1, 45.8, 47.0, 46.1, 45.9, 45.8, 46.9, 45.2, and 46.0. Find a 95% confidence interval for the variance of the weights of all such packages of grass seed distributed by this company, assuming a normal population.